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Derivation and solution of the two-dimensional Toda lattice equations by use of the Iwasawa decomposition

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Abstract. The Iwasawa decomposition is applied to the complexification of an arbitrary gauge group G over a two-dimensional space. The two-dimensional Toda lattice equations arise from the condition that the gauge field strength vanishes. The method is based on the generalisation of Yang's R-gauge for SU(2) to any semi-simple Lie group G. A parametrisation for the solutions to the equations governed by arbitrary classical groups is obtained.

1. Introduction

Recent years have seen the generation of much interest in the nonlinear lattice equations, discovered by Toda in 1966, which describe the motion of a one-dimensional system of particles. When the only interactions are between neighbouring particles and these have an exponential form, exact solutions of the equations have been found (Toda 1967). The surge of interest is due to the recognition that a rich mathematical structure, both algebraic and geometric, is associated with the equations‡.

It is the association of the equations with the theory of groups and their corresponding Lie algebras which concerns us in this paper, so we shall briefly review this aspect. The equations of motion for a finite system of $(n+1)$ particles arranged in line are

$$\partial_t^2 \rho_i = -\sum_{j=1}^{n} K_{ij} \exp \rho_j, \tag{1.1}$$

where ρ_i is the difference in the displacements of the $(i+1)$th and the ith particle. Toda's original model corresponds to the $n = \infty$ limit of this finite system. In (1.1), K is an $n \times n$ matrix with the only non-zero entries as follows:

$$K_{ii} = 2, \qquad i = 1, 2, \ldots, n,$$
$$K_{i+1\,i} = K_{i\,i+1} = -1, \qquad i = 1, 2, \ldots, n-1.$$

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‡ Some of the more seminal contributions to the mathematical structure were: discovery of Lax pair; derivation of equations for finite systems and their solution; analogy with group structure (Lax 1968, Flaschka 1974a, b, Manakov 1974, Bogoyavlensky 1976, etc). Subsequently, Leznov and Saveliev (1978) in considering the two-dimensional model used the connection with the Cartan matrix; Olshanetsky and Perelomov (1979) couched the solution to the one-dimensional model in algebraic terminology, and independently Kostant (1979) found solutions to more general integrable systems.

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However, this matrix arises in the theory of Lie groups as the Cartan matrix for SU$(n+1)$. Furthermore, the Dynkin diagram for the corresponding Lie algebra pictorially resembles the system of particles interacting only with their neighbours. It is now apparent that the model may be generalised to include systems described by (1.1), but where (K_{ij}) is the Cartan matrix for any semi-simple group. It has subsequently been realised that the integrability and solubility properties of the Toda lattice equations will also apply to the generalised equations.

Toda's model is not only of interest mathematically, since it also has some physical applications. There are similarities between the lattice and its continuum approximation, which is described by the Korteweg–de Vries equation and which is studied in plasma physics. Other applications are found in gauge field theories. The simplest Toda lattice equation with $n=1$ and governed by SU(2) has the same form as the Liouville equation, which has appeared in SU(2) gauge theories. In Euclidean space, SU(2) instantons with cylindrical symmetry are solutions of this equation (Witten 1977), whilst in a broken SU(2) theory the only time-independent solution corresponds to the Prasad–Sommerfield monopole solution (Bais and Weldon 1978). Also in the string model, Omnés claims that the classical states correspond to solutions of the same equation (Omnés 1979).

The most recent development, and the one which has motivated our study, is the discovery that the Bogomolny–Prasad–Sommerfield equation for the spherically symmetric monopole for any group may also be written in the form of the Toda lattice equation (Olive 1980). The effective generalisation of the Toda model to the other groups has therefore suggested a way in which monopoles may be described in theories with an arbitrary semi-simple gauge group.

Our discussion thus far has been restricted to one-dimensional SU$(n+1)$ Toda lattices and also their generalisation to one-dimensional models with any semi-simple compact group. However, other models may be considered; for example, the one-dimensional periodic Toda lattice represented by the extended Dynkin diagram and also two-dimensional models with one time and one space variable.

Leznov and Saveliev (1978, 1979a, b, 1980) have developed the theory of general lattices composed of a finite non-periodic chain in two time dimensions. The equations of motion for this system are given by

$$(\partial_t^2+\partial_s^2)\rho_i=-4\sum_j K_{ij}\exp\rho_j.$$

Leznov and Saveliev have remarked that when (K_{ij}) is a 2nd-order generalised Cartan matrix for an infinite-dimensional contragredient Lie algebra, this becomes the sinh-Gordon equation.

Our particular study is concerned with similar non-periodic lattices and so with an arbitrary semi-simple group of finite rank. We specifically consider Euclidean space with the two variables (s, t) and employ complex coordinates $u=s+\mathrm{i}t$ and $\bar{u}=s-\mathrm{i}t$. The generalised Toda lattice equations become

$$\partial_u\partial_{\bar{u}}\rho_i=-\sum_j K_{ij}\exp\rho_j \tag{1.2}$$

or alternatively, defining

$$\rho_j=-\sum_i K_{ji}\psi_i, \tag{1.3}$$

$$\partial_u \partial_{\bar{u}} \psi_i = \exp\Big(-\sum_j K_{ij}\psi_j\Big). \tag{1.4}$$

As the variables are complexified, the Lie group G governing the equation must also be complexified to

$$G_c = G \times iG.$$

As a consequence of the ensuing non-compactness of G_c, we can apply the Iwasawa decomposition (Helgason 1978, Hermann 1966) to it in such a way that the origin of the Toda lattice variables will become apparent. It also leads to an easy means of parametrising the solutions of the equation.

Our method of deriving the Toda lattice equations is somewhat heuristic and so, to demonstrate its plausibility and neatness, we shall explain it here with reference to Yang's approach to the self-duality condition for SU(2) gauge fields. This approach is known as Yang's R-gauge method and is itself an application of the Iwasawa decomposition (Yang 1977, hereafter referred to as Y).

On four-dimensional Euclidean space with coordinates $\{x_\mu, \mu = 1, 2, 3, 4\}$ the self-duality condition for the gauge field strength $F_{\mu\nu}$ is

$$F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{1.5}$$

where $F_{\mu\nu}$ is written in terms of the gauge potentials B_μ by

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]. \tag{1.6}$$

By complexifying the coordinates and defining four new variables

$$\begin{aligned} \sqrt{2}y &= x_1 + ix_2, \qquad & \sqrt{2}\bar{y} &= x_1 - ix_2, \\ \sqrt{2}z &= x_3 - ix_4, \qquad & \sqrt{2}\bar{z} &= x_3 + ix_4, \end{aligned} \tag{1.7}$$

and the corresponding components of the potentials, Yang is able to rewrite (1.5) in the form

$$F_{yz} = F_{\bar{y}\bar{z}} = 0, \tag{1.8}$$

$$F_{y\bar{y}} + F_{z\bar{z}} = 0. \tag{1.9}$$

Equations (1.8) and (1.9) provide a means of determining B_μ: indeed, (1.8) imply that

$$\begin{aligned} B_y &= R^{-1}\partial_y R, \qquad & B_z &= R^{-1}\partial_z R, \\ B_{\bar{y}} &= \bar{R}^{-1}\partial_{\bar{y}}\bar{R}, \qquad & B_{\bar{z}} &= \bar{R}^{-1}\partial_{\bar{z}}\bar{R}, \end{aligned} \tag{1.10}$$

where $\det R = \det \bar{R} = 1$. $\bar{R}$ is defined such that in the real section

$$\bar{R} = (R^\dagger)^{-1}. \tag{1.11}$$

By making a gauge transformation, Yang argues that it is always possible to choose R to have the lower triangular form

$$R = \frac{1}{\sqrt{\phi}}\begin{pmatrix} 1 & 0 \\ \rho & \phi \end{pmatrix}, \tag{1.12}$$

where ϕ is a real function and ρ is a complex function on the real section. By using the relation (1.11), $\bar{R}$ is seen to have an upper triangular form. The resultant gauge is known as the R-gauge.

By substituting (1.10) into (1.9) when R is given by the expression (1.12), the equation (1.9) simplifies (see equation (27) in Y) and the problem of determining B_μ is reduced to one of solving this simplified form for ρ and ϕ.

Our first observation is that in the sector $y = z \equiv u$, Yang's equation (1.9) (and so equation (27) in Y) becomes over the real section

$$\phi\partial_u\partial_{\bar u}\phi - (\partial_u\phi)(\partial_{\bar u}\phi) + (\partial_u\rho)(\partial_{\bar u}\bar\rho) = 0, \tag{1.13}$$

$$\begin{aligned} &\phi\partial_{\bar u}(\partial_u\rho) - 2(\partial_u\rho)\partial_{\bar u}\phi = 0,\\ &\phi\partial_u(\partial_{\bar u}\bar\rho) - 2(\partial_{\bar u}\bar\rho)\partial_u\phi = 0. \end{aligned} \tag{1.14}$$

A notable fact is that (1.14) (hereafter called the subsidiary equations) are soluble. Indeed, if we let

$$\phi = \exp(-2\psi), \tag{1.15}$$

then the solutions are

$$\partial_u\rho = \varepsilon(u)\,e^{-4\psi}, \qquad \partial_{\bar u}\bar\rho = \bar\varepsilon(\bar u)\,e^{-4\psi}. \tag{1.16}$$

By putting (1.16) into (1.13), we obtain

$$\partial_u\partial_{\bar u}\psi = \varepsilon\bar\varepsilon \exp(-4\psi). \tag{1.17}$$

If we choose $\varepsilon\bar\varepsilon = 1$ in (1.17), comparison of the resultant equation with (1.4) shows that we have obtained the simplest Toda lattice equation governed by G = SU(2).

In the sector $y = z$, (1.8) are automatically satisfied and (1.9) becomes

$$F_{u\bar u} \equiv \partial_u B_{\bar u} - \partial_{\bar u}B_u + [B_u, B_{\bar u}] = 0. \tag{1.18}$$

We have shown that when B_u and $B_{\bar u}$ are given by

$$B_u = R^{-1}\partial_u R, \qquad B_{\bar u} = \bar R^{-1}\partial_{\bar u}\bar R, \tag{1.19}$$

the equation (1.18) reduces to the Toda lattice equation.

Our derivation in § 2 of the Toda lattice equations using the Iwasawa decomposition is based upon the example above. We show that the same argument may be applied even when the group governing the equation is extended to any semi-simple Lie group. The method also suggests a means of solving the equations governed by any classical group and this is presented in § 3, together with some specific examples. The general parametrisation of the SU($n+1$) solution is included in appendix 2. In § 4 we make some concluding remarks about the reality conditions on B_u and $B_{\bar u}$ and also the similarity between the one-dimensional Toda model and our two-dimensional model.

2. Iwasawa decomposition and the Toda lattice equations

We first note that Yang's R-matrix (1.12), having complex entries and unit determinant, belongs to the group SL(2, ℂ), which is the complexification of the original gauge group, SU(2). Furthermore R has a lower triangular form with real diagonal entries. It is possible to show that such a form is obtained by applying the Iwasawa decomposition NAK to SL(2, ℂ), where K is the maximal compact subgroup and so is SU(2) in this case (Ardalan 1978). In particular, by choosing a gauge so that the element belonging to K in the decomposition is effectively unity, the explicit form (1.12) is obtained. Having observed these properties of the R-gauge, it is obvious that the method previously outlined in the introduction of deriving the Toda lattice equations governed by SU(2) can be extended to derive the equations governed by $SU(n+1)$. The Iwasawa decomposition applied to $SL(n+1, \mathbb{C})$ specifies an $SU(n+1)$ gauge in which the gauge potentials have triangular form with real diagonal elements. For this reason, Yang's R-gauge and its generalisation to $SU(n+1)$ have been referred to as the triangular gauge (Brihaye *et al* 1978).

We follow Ardalan (1978) and obtain the analogue of the R-gauge not only when the gauge group $G = SU(n+1)$, but also when G is any compact semi-simple group. However, we choose representations such that R is not always triangular. Our intention is to derive the Toda lattice equations governed by an arbitrary group and then find solutions of the equation for all the groups associated with the classical simple Lie algebras.

We must first complexify the group G to G_c and then invoke the corresponding complex-valued Lie algebra. The group G may either be classical or exceptional and the former class may be divided into the following subclasses: $SU(n+1)$, $SO(2n+1)$, $Sp(n)$ and $SO(2n)$†. It should be noted that in all cases the maximal compact subgroup of the complex group G_c coincides with G (Gilmore 1974). If $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$, then it may be decomposed according to Cartan's root space decomposition (Helgason 1978) as

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta^+} (\mathfrak{g}_{+\alpha} + \mathfrak{g}_{-\alpha}), \tag{2.1}$$

where $\mathfrak{h}$ is a fixed Cartan subalgebra, Δ^+ is the positive root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $\mathfrak{g}_{\pm\alpha}$ are the root spaces defined by

$$\mathfrak{g}_{\pm\alpha} = \{E \in \mathfrak{g}\colon \operatorname{ad}(H)E = \pm\alpha(H)E \text{ for all } H \in \mathfrak{h}\}.$$

We shall use the Chevalley basis $(H_\alpha, E_{\pm\alpha})$ for $\mathfrak{g}$ (Humphreys 1972, Carter 1972). H_α is a basis for $\mathfrak{h}$ and $E_{\pm\alpha}$ a basis for $\mathfrak{g}_{\pm\alpha}$, such that

$$[H_\alpha, H_\beta] = 0, \tag{2.2a}$$

$$[H_\alpha, E_{\pm\beta}] = \pm K_{\beta\alpha} E_{\pm\beta}, \tag{2.2b}$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \tag{2.2c}$$

$$[E_\alpha, E_\beta] = \begin{cases} 0 & \alpha+\beta \notin \Delta, \alpha \neq \beta, \\ N_{\alpha,\beta} E_{\alpha+\beta} & \text{otherwise}, \alpha \neq \beta. \end{cases} \tag{2.2d}$$

Only n, where $n = \operatorname{rank} \mathfrak{g}$, of the H's are linearly independent and the corresponding set of n positive roots is denoted by π^+. Any root $\alpha \in \pi^+$ is known as a simple root, that is, it cannot be written as the sum of other positive roots. As a consequence of this property, we can show that for $\alpha, \beta \in \pi^+$

$$N_{\alpha,\beta} = N_{\alpha,-\beta} = 0.$$

In addition for α, β simple, the matrix $(K_{\beta\alpha})$ defined by the commutation relation (2.2b) is the $(n \times n)$ Cartan matrix associated with $\mathfrak{g}$.

Since $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$, we may consider it as a Lie algebra $\mathfrak{g}^R$ over $\mathbb{R}$ with complex structure, and the associated Lie group may be denoted by G_c. By

† Complex extensions of $SU(n+1)$, $SO(2n+1)$, $Sp(n)$ and $SO(2n)$ are denoted $SL(n+1, \mathbb{C})$, $SO(2n+1, \mathbb{C})$, $Sp(n, \mathbb{C})$ and $SO(2n, \mathbb{C})$ and their algebras $\mathfrak{a}_n$, $\mathfrak{b}_n$, $\mathfrak{c}_n$ and $\mathfrak{d}_n$ respectively. It should be remarked that $Sp(n)$ is not $Sp(n, R)$, the latter not being compact.

combining the Cartan decomposition of $\mathfrak{g}$ and the root space decomposition of $\mathfrak{g}^R$, the Iwasawa decomposition for G_c is obtained. It is given by

$$G_c = NAK, \tag{2.3}$$

where N and A are subgroups of G_c corresponding to the subalgebras $\mathfrak{g}_{-\alpha}$ and $\mathfrak{h}$ respectively, and K is the maximal compact subgroup. The decomposition NAK is quite general and may be applied to any non-compact, semi-simple Lie group. However, if G_c is of the form

$$G_c = iG \times G, \tag{2.4}$$

we may be more specific about N, A and K. Indeed,

$$K = G,$$

the original gauge group; also A is generated by $\mathfrak{h}$ over $\mathbb{R}$ and N by $\mathfrak{g}_{-\alpha}$ over $\mathbb{C}$. Hence, any group element $g \in G_c$ may be written as

$$g = nak \tag{2.5}$$

with $n \in N$, $a \in A$, $k \in G$. n and a are given by

$$n = \exp\Big(\sum_{\alpha \in \Delta^+} z_\alpha E_{-\alpha}\Big), \qquad a = \exp\Big(\sum_{\alpha \in \pi^+} \psi_\alpha H_\alpha\Big), \tag{2.6}$$

where z_α are complex functions and ψ_α are real functions, all dependent on both coordinates, u and $\bar{u}$,

$$u = s + it, \qquad \bar{u} = s - it. \tag{2.7}$$

Note that the suffix α of H_α corresponds to the simple roots, so only the linearly independent basis elements are included.

Our discussion concerning the Iwasawa decomposition has involved the subgroup of G_c corresponding to the nilpotent subalgebra $\mathfrak{g}_{-\alpha}$, but not the subgroup corresponding to the equivalent subalgebra $\mathfrak{g}_{+\alpha}$, which is spanned by $E_{+\alpha}$. It is possible to include this other subgroup in another version of the Iwasawa decomposition in which any element $\tilde{g} \in G_c$ is given by

$$\tilde{g} = \tilde{n}\tilde{a}\tilde{k}. \tag{2.8}$$

In (2.8) $\tilde{k} \in G$ and the representations for $\tilde{n}$ and $\tilde{a}$ are

$$\tilde{n} = \exp\Big(-\sum_{\alpha \in \Delta^+} \tilde{z}_\alpha E_{+\alpha}\Big), \qquad \tilde{a} = \exp\Big(-\sum_{\alpha \in \pi^+} \tilde{\psi}_\alpha H_\alpha\Big). \tag{2.9}$$

To generalise the R-gauge method, we gauge away the elements $k, \tilde{k} \in G$ in the decompositions (2.5) and (2.8) for g and $\tilde{g}$ respectively. This depends on choosing the same gauge for both elements, g and $\tilde{g}$, and so we restrict them by imposing the condition

$$k = \tilde{k}. \tag{2.10}$$

However, it should be noted that to derive the Toda lattice equations it is not necessary to assume any relationship between the parameters z_α, ψ_α in the representation of na and $\tilde{z}_\alpha$, $\tilde{\psi}_\alpha$ in $\tilde{n}\tilde{a}$.

The condition (2.10) allows us to choose the gauge in which g and $\tilde{g}$ are given by R and $\tilde{R}$, where

$$R = na, \qquad \tilde{R} = \tilde{n}\tilde{a}. \tag{2.11}$$

By analogy with (1.19) we define the left-invariant gauge potentials B_u and $B_{\bar{u}}$ by

$$B_u(u, \bar{u}) = R^{-1}\partial_u R, \qquad B_{\bar{u}}(u, \bar{u}) = \tilde{R}^{-1}\partial_{\bar{u}}\tilde{R}. \tag{2.12}$$

After substituting the explicit forms (2.11) for R and $\tilde{R}$ in (2.12), we obtain

$$B_u = a^{-1}(n^{-1}\partial_u n)a + a^{-1}\partial_u a, \qquad B_{\bar{u}} = \tilde{a}^{-1}(\tilde{n}^{-1}\partial_{\bar{u}}\tilde{n})\tilde{a} + \tilde{a}^{-1}\partial_{\bar{u}}\tilde{a}. \tag{2.13}$$

We now adopt the following ansatz:

$$n^{-1}\partial_u n = \sum_{\alpha\in\pi^+} y_\alpha E_{-\alpha}, \qquad \tilde{n}^{-1}\partial_{\bar{u}}\tilde{n} = -\sum_{\alpha\in\pi^+} \tilde{y}_\alpha E_{+\alpha}, \tag{2.14}$$

which is equivalent to the assumption that, when α is not a simple root,

$$y_\alpha = \tilde{y}_\alpha = 0, \tag{2.15}$$

that is, for $\alpha \notin \pi^+$. This restriction to include only the simple roots is in accordance with the Toda model developed from the theory of Lax pairs (Bogoyavlensky 1976) in which the non-simple roots do not play an important role.

By using the representations for n and $\tilde{n}$ in (2.6) and (2.9) and the commutation relations (2.2d), we can show that for $\alpha \in \pi^+$

$$y_\alpha = \partial_u z_\alpha, \qquad \tilde{y}_\alpha = \partial_{\bar{u}}\tilde{z}_\alpha. \tag{2.16}$$

Also from the representations for a and $\tilde{a}$, it is straightforward to obtain

$$a^{-1}\partial_u a = \sum_{\alpha\in\pi^+} (\partial_u\psi_\alpha)H_\alpha, \qquad \tilde{a}^{-1}\partial_{\bar{u}}\tilde{a} = -\sum_{\alpha\in\pi^+} (\partial_{\bar{u}}\tilde{\psi}_\alpha)H_\alpha \tag{2.17}$$

due to the commutation of the basis elements H_α. Further,

$$a^{-1}E_{-\alpha}a = \exp\Big(\sum_{\beta\in\pi^+} K_{\alpha\beta}\psi_\beta\Big)E_{-\alpha}, \qquad \tilde{a}^{-1}E_{+\alpha}\tilde{a} = \exp\Big(\sum_{\beta\in\pi^+} K_{\alpha\beta}\tilde{\psi}_\beta\Big)E_{+\alpha} \tag{2.18}$$

and so by combining (2.14)–(2.18) with (2.13), we have

$$\begin{aligned} B_u &= \sum_{\alpha\in\pi^+}\Big[y_\alpha \exp\Big(\sum_{\beta\in\pi^+} K_{\alpha\beta}\psi_\beta\Big)E_{-\alpha} + (\partial_u\psi_\alpha)H_\alpha\Big], \\ B_{\bar{u}} &= -\sum_{\alpha\in\pi^+}\Big[\tilde{y}_\alpha \exp\Big(\sum_{\beta\in\pi^+} K_{\alpha\beta}\tilde{\psi}_\beta\Big)E_{+\alpha} + (\partial_{\bar{u}}\tilde{\psi}_\alpha)H_\alpha\Big]. \end{aligned} \tag{2.19}$$

In order to impose the condition (1.18) that

$$F_{u\bar{u}} = \partial_u B_{\bar{u}} - \partial_{\bar{u}} B_u + [B_u, B_{\bar{u}}]$$

vanishes, we calculate firstly

$$\begin{aligned} [B_u, B_{\bar{u}}] = &-\sum_{\alpha,\,\gamma\in\pi^+}\Big[y_\alpha(\partial_{\bar{u}}\tilde{\psi}_\gamma)\exp\Big(\sum_{\beta\in\pi^+} K_{\alpha\beta}\psi_\beta\Big)K_{\alpha\gamma}E_{-\alpha} \\ &+ \tilde{y}_\alpha(\partial_u\psi_\gamma)\exp\Big(\sum_{\beta\in\pi^+} K_{\alpha\beta}\tilde{\psi}_\beta\Big)K_{\alpha\gamma}E_{+\alpha}\Big] \\ &+ \sum_{\alpha\in\pi^+} y_\alpha\tilde{y}_\alpha \exp\Big(\sum_{\beta\in\pi^+} K_{\alpha\beta}(\psi_\beta + \tilde{\psi}_\beta)\Big)H_\alpha, \end{aligned} \tag{2.20}$$

and secondly

$$\partial_{\bar u}B_u-\partial_u B_{\bar u}=\sum_{\alpha\in\pi^+}\Big[\Big(\partial_{\bar u}y_\alpha+y_\alpha\sum_{\beta\in\pi^+}K_{\alpha\beta}\partial_{\bar u}\psi_\beta\Big)\exp\Big(\sum_{\beta\in\pi^+}K_{\alpha\beta}\psi_\beta\Big)E_{-\alpha}+(\partial_{\bar u}\partial_u\psi_\alpha)H_\alpha$$
$$+\Big(\partial_u\tilde y_\alpha+\tilde y_\alpha\sum_{\beta\in\pi^+}K_{\alpha\beta}\partial_u\tilde\psi_\beta\Big)\exp\Big(\sum_{\beta\in\pi^+}K_{\alpha\beta}\tilde\psi_\beta\Big)E_{+\alpha}+(\partial_u\partial_{\bar u}\tilde\psi_\alpha)H_\alpha\Big].\tag{2.21}$$

Now by equating the coefficients of H_α and $E_{\pm\alpha}$ in (2.20) and (2.21) the condition reduces to the following three equations:

$$y_\alpha\tilde y_\alpha\exp\Big(\sum_{\beta\in\pi^+}K_{\alpha\beta}(\psi_\beta+\tilde\psi_\beta)\Big)=\partial_u\partial_{\bar u}(\psi_\alpha+\tilde\psi_\alpha),\tag{2.22}$$

$$\begin{aligned}\partial_{\bar u}y_\alpha+y_\alpha\sum_{\beta\in\pi^-}K_{\alpha\beta}\partial_{\bar u}\psi_\beta&=-y_\alpha\sum_{\beta\in\pi^+}K_{\alpha\beta}(\partial_{\bar u}\tilde\psi_\beta),\\ \partial_u\tilde y_\alpha+\tilde y_\alpha\sum_{\beta\in\pi^+}K_{\alpha\beta}\partial_u\tilde\psi_\beta&=-\tilde y_\alpha\sum_{\beta\in\pi^+}K_{\alpha\beta}(\partial_u\psi_\beta).\end{aligned}\tag{2.23}$$

We note that for $G=SU(2)$, there is only one simple root and $K=2$, so in this case (2.22) and (2.23) correspond to (1.13) and (1.14) respectively with

$$y=\partial_u\rho,\qquad \tilde y=\partial_{\bar u}\bar\rho,\qquad \phi=e^{-(\psi+\tilde\psi)}.\tag{2.24}$$

Therefore, by analogy (2.22) is the main equation and (2.23) are the subsidiary equations. Fortunately, as in the SU(2) case, we are able to solve the subsidiary equations to give

$$y_\alpha=\varepsilon(u)\exp\Big(-\sum_{\beta\in\pi^+}K_{\alpha\beta}(\psi_\beta+\tilde\psi_\beta)\Big),\qquad \tilde y_\alpha=\tilde\varepsilon(\bar u)\exp\Big(-\sum_{\beta\in\pi^+}K_{\alpha\beta}(\psi_\beta+\tilde\psi_\beta)\Big),\tag{2.25}$$

where $\varepsilon(u)$ and $\tilde\varepsilon(\bar u)$ are to be determined by the 'boundary conditions'. If we put the solutions (2.25) into (2.22), then the main equation becomes

$$\partial_u\partial_{\bar u}(\psi_\alpha+\tilde\psi_\alpha)=\varepsilon\tilde\varepsilon\exp\Big(-\sum_{\beta\in\pi^+}K_{\alpha\beta}(\psi_\beta+\tilde\psi_\beta)\Big).\tag{2.26}$$

When $\varepsilon\tilde\varepsilon=1$, the equation (2.26) is one particular form of the Toda lattice equation governed by an arbitrary group G with Cartan matrix K. The form (1.2) of the Toda lattice equation may be derived by defining the new variables

$$\sigma_\alpha=-\sum_{\beta\in\pi^+}K_{\alpha\beta}\psi_\beta,\qquad \tilde\sigma_\alpha=-\sum_{\beta\in\pi^+}K_{\alpha\beta}\tilde\psi_\beta\tag{2.27a}$$

and

$$\rho_\alpha=\sigma_\alpha+\tilde\sigma_\alpha.\tag{2.27b}$$

However, it should be said that the algebraic meaning of the variables ρ_α is somewhat vague, but in our form (2.26) we can trace the origin of the ψ_α and $\tilde\psi_\alpha$ back to the coefficients of the generators of the abelian subgroup in the Iwasawa decomposition.

Also we can give explicit expressions for the potentials B_u and $B_{\bar{u}}$ by substituting the solutions (2.25) for y_α and $\tilde{y}_\alpha$ into (2.19). In particular, when $\varepsilon = \tilde{\varepsilon} = 1$, this gives

$$\begin{aligned} B_u &= \sum_{\alpha\in\pi^+}\left[\exp\left(-\sum_{\beta\in\pi^+} K_{\alpha\beta}\tilde{\psi}_\beta\right)E_{-\alpha} + (\partial_u\psi_\alpha)H_\alpha\right], \\ B_{\bar{u}} &= -\sum_{\alpha\in\pi^+}\left[\exp\left(-\sum_{\beta\in\pi^+} K_{\alpha\beta}\psi_\beta\right)E_{+\alpha} + (\partial_{\bar{u}}\tilde{\psi}_\alpha)H_\alpha\right]. \end{aligned} \tag{2.28}$$

We close this section by remarking that indeed the Iwasawa decomposition has allowed us to generalise Yang's R-gauge method and hence obtain the Toda lattice equations governed by an arbitrary semi-simple Lie group. We reiterate that to derive these equations it is not necessary to impose any condition relating the parameters of R and $\tilde{R}$. This differs from the R-gauge where from (2.24) we see that in the real section $\tilde{y} = \bar{y}$, where the bar denotes complex conjugation, and, for consistency with (1.15), $\tilde{\psi} = \psi$. We discuss this point in greater detail in a later section.

3. Parametrisation of the solutions

The purpose of this section is to demonstrate a method of obtaining the solutions to the Toda lattice equations (2.26) when the governing group G is classical. The solutions are parametrised in the form of determinants. As in § 2, it is the Iwasawa decomposition of R and $\tilde{R}$ given by (2.11) which plays an important role.

Since we have factored out the gauge terms in g and $\tilde{g}$, the resultant elements R and $\tilde{R}$ must belong to the symmetric space G_c/G. In addition B_u and $B_{\bar{u}}$ given by (2.19) or (2.28) compose a matrix connection one-form, taking values in the algebra corresponding to G_c, which is defined by

$$\omega = B_u \mathrm{d}u + B_{\bar{u}}\, \mathrm{d}\bar{u}. \tag{3.1}$$

Then the condition (1.18) implies that the curvature two-form must vanish, that is,

$$\Omega \equiv \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega] = 0. \tag{3.2}$$

Conversely, if $\Omega = 0$, then the connection ω should be a left-invariant one-form given by

$$\omega = g_0^{-1}\, \mathrm{d}g_0 \tag{3.3}$$

where $g_0 \in G_c$. Consequently B_u and $B_{\bar{u}}$ are given by

$$B_u = g_0^{-1}\,\partial_u g_0, \qquad B_{\bar{u}} = g_0^{-1}\,\partial_{\bar{u}} g_0. \tag{3.4}$$

However, we have already capitalised on the definitions of B_u and $B_{\bar{u}}$ given by (2.12). To show that the two sets of definitions (3.4) and (2.12) are non-contradictory, we make explicit the property alluded to in § 2 that the gauge potentials are left-invariant. In fact, B_u and $B_{\bar{u}}$ given by (3.4) are unchanged when g_0 is transformed by the respective left-translations

$$g_0 \to \tilde{r}^{-1}(\bar{u})g_0, \qquad g_0 \to r^{-1}(u)g_0. \tag{3.5}$$

Hence, if we are permitted to make the identifications

$$R = \tilde{r}^{-1}(\bar{u})g_0, \qquad \tilde{R} = r^{-1}(u)g_0, \tag{3.6}$$

at least up to a gauge transformation, then the two definitions are consistent. In general

r and $\tilde{r}$ may belong to G_c, but are restricted so that r and $\tilde{r}$ depend only on u and $\bar{u}$ respectively. We use (2.3) to write $\tilde{r}$ and r in decomposed form as

$$\tilde{r} = \tilde{n}_0 \tilde{a}_0 \tilde{k}_0, \qquad r = n_0 a_0 k_0.$$

By use of these equations, (3.6) can be rearranged to give

$$g_0 = \tilde{n}_0 \tilde{a}_0 \tilde{k}_0 R, \qquad g_0 = n_0 a_0 k_0 \tilde{R}. \tag{3.7}$$

It is known that for all $k'' \in G$, Ad k'' leaves G_c invariant. In particular (Hermann 1966, Kostant 1973)

$$k'' n a k''^{-1} = n' a' k',$$

where $n' \in N$, $a' \in A$, $k' \in K$, and this may be rewritten as

$$k'' n a = n' a' k' k''. \tag{3.8}$$

We use (3.8) to rearrange equations (3.7) to give

$$g_0 = \tilde{n}_0 \tilde{a}_0 R' k' \tilde{k}_0, \tag{3.9a}$$

$$g_0 = n_0 a_0 \tilde{R}' \tilde{k}' k_0, \tag{3.9b}$$

where

$$R' = n' a', \qquad \tilde{R}' = \tilde{n}' \tilde{a}'.$$

We now require that R and $\tilde{R}$ of § 2 be replaced by R' and $\tilde{R}'$. If we define B'_u and $B'_{\bar{u}}$ from R' and $\tilde{R}'$ respectively by analogy with (2.12), then these are just gauge transformations of the potentials B_u and $B_{\bar{u}}$ given by (3.4). For consistency, B'_u and $B'_{\bar{u}}$ must be defined in the same gauge, and hence in (3.9) we put

$$k' \tilde{k}_0 = \tilde{k}' k_0. \tag{3.10}$$

By equating the right-hand sides of (3.9) and imposing the condition (3.10), we obtain

$$\tilde{a}_0 R \tilde{R}^{-1} a_0^{-1} = \tilde{n}_0^{-1} n_0, \tag{3.11}$$

where we have dropped the primed notation. Hence (3.11) may be obtained effectively from (3.6) when $\tilde{r}$ and r are given by†

$$\tilde{r} = \tilde{n}_0(\bar{u}) \tilde{a}_0(\bar{u}), \qquad r = n_0(u) a_0(u). \tag{3.12}$$

We use the following representations for the decompositions (3.12):

$$\begin{aligned} a_0 &= \exp\Big(\sum_{\alpha \in \pi^+} \mu_\alpha(u) H_\alpha\Big), & \tilde{a}_0 &= \exp\Big(-\sum_{\alpha \in \pi^+} \tilde{\mu}_\alpha(\bar{u}) H_\alpha\Big), \\ n_0 &= \exp\Big(\sum_{\alpha \in \Delta^+} \nu_\alpha(u) E_{-\alpha}\Big), & \tilde{n}_0 &= \exp\Big(-\sum_{\alpha \in \Delta^+} \tilde{\nu}_\alpha(\bar{u}) E_{+\alpha}\Big). \end{aligned} \tag{3.13}$$

Our method of solution depends on taking determinants of submatrices of the left- and right-hand side of (3.11) and then equating. Firstly, we consider the left-hand side,

† David Olive has made us aware that these restricted forms for r and $\tilde{r}$ may be ascribed to the path independency of $g_0 = T \exp \int_{(0,0)}^{(u,\bar{u})} (B_u \, du + B_{\bar{u}} \, d\bar{u})$, Since $R(u, \bar{u}) = T \exp \int_{(0,\bar{u})}^{(u,\bar{u})} B_u \, du$ and $\tilde{R}(u, \bar{u}) = T \exp \int_{(u,0)}^{(u,\bar{u})} B_{\bar{u}} \, d\bar{u}$, we can write g_0 in two ways: $g_0 = R(u, 0) \tilde{R}(u, \bar{u}) = \tilde{R}(0, \bar{u}) R(u, \bar{u})$. We thank him for this communication.

which is explicitly given by

$$L = \exp(-\tilde{\mu}_\alpha H_\alpha)\exp(z_\beta E_{-\beta})\exp[(\psi_\alpha + \tilde{\psi}_\alpha)H_\alpha]\exp(\tilde{z}_\beta E_{+\beta})\exp(-\mu_\alpha H_\alpha), \tag{3.14}$$

where in the exponents the index β is summed over all positive roots, while the index α is summed only over the simple roots. Since for all the classical algebras the H_α are diagonal, the properties of L are essentially those of the product of $\exp(z_\beta E_{-\beta})$ and $\exp(\tilde{z}_\beta E_{+\beta})$. Now if L' is an $(n \times n)$ lower triangular matrix and L'' is an $(n \times n)$ upper triangular matrix with

$$L = L'L'',$$

then the only contributions to the $(m \times m)$ submatrix $(L_{kl})_{1 \leqslant k,l \leqslant m}$ are from the $(m \times m)$ submatrices $(L'_{kl})_{1 \leqslant k,l \leqslant m}$ and $(L''_{kl})_{1 \leqslant k,l \leqslant m}$. Hence, for $1 \leqslant k,\ l \leqslant m$,

$$\det(L_{kl}) = \det(L'_{kl})\det(L''_{kl}). \tag{3.15}$$

In appendix 1 for each classical Lie algebra we specify some set of submatrices $\{((E_{-\alpha})_{lm}):\ l \leqslant m\}$ in which each member has a lower triangular matrix lying in a block where all the other entries are zero. We restrict our attention to these submatrices, and then from (3.14), we can show that by using (3.15)

$$\begin{aligned}\det(L_{kl}) = {} & \det[(\exp(-\tilde{\mu}_\alpha H_\alpha))_{kl}]\det[(\exp(z_\alpha E_{-\alpha}))_{kl}]\det[(\exp[(\psi_\alpha + \tilde{\psi}_\alpha)H_\alpha])_{kl}] \\ & \times \det[(\exp(\tilde{z}_\alpha E_{+\alpha}))_{kl}]\det[(\exp(-\mu_\alpha H_\alpha))_{kl}]\end{aligned}$$

where $1 \leqslant k,\ l \leqslant m \leqslant n$ for $\mathfrak{a}_n$, $\mathfrak{c}_n$, $\mathfrak{d}_n$, $2 \leqslant k, l \leqslant m \leqslant n+1$ for $\mathfrak{b}_n$.

Since for each algebra in the appropriate range

$$\det[(\exp(z_\alpha E_{-\alpha}))_{kl}] = 1,$$

and since the H_α are diagonal, this becomes

$$\det(L_{kl}) = \exp\{\mathrm{Tr}[(-\tilde{\mu}_\alpha H_\alpha)_{kl}]\}\exp\{\mathrm{Tr}[((\psi_\alpha - \tilde{\psi}_\alpha)H_\alpha)_{kl}]\}\exp\{\mathrm{Tr}[(-\mu_\alpha H_\alpha)_{kl}]\}.$$

Therefore, (3.13) may be written as

$$\exp\{\mathrm{Tr}[((\psi_\alpha + \tilde{\psi}_\alpha)H_\alpha)_{kl}]\} = \exp\{\mathrm{Tr}[((\mu_\alpha + \tilde{\mu}_\alpha)H_\alpha)_{kl}]\}\det[(\tilde{n}_0^{-1}n_0)_{kl}]. \tag{3.16}$$

Before reducing this expression further, we shall consider the representations (3.12) for a_0, $\tilde{a}_0$, n_0 and $\tilde{n}_0$. (3.11) provides a relationship between R and $\tilde{R}$,

$$R = \tilde{r}^{-1} r \tilde{R}$$

and so B_u, for instance, is given by

$$B_u = \tilde{R}^{-1} r^{-1}(\partial_u r)\tilde{R} + \tilde{R}^{-1}\partial_u \tilde{R}, \tag{3.17}$$

as well as by (2.12). In (2.13) B_u has a specific form depending only on H_α and $E_{-\alpha}$ and, moreover, as a consequence of the ansatz (2.14), the basis elements correspond only to the simple roots. Hence there must be some condition on the parameters in r so that B_u given by (2.17) also has these properties. In particular, we are concerned with the coefficient of $E_{-\alpha}$ in (3.17). By using (3.12),

$$r^{-1}\partial_u r = a_0^{-1}(n_0^{-1}\partial_u n_0)a_0 + a_0^{-1}\partial_u a_0$$

and so, by referring to the commutation relations (2.2), we see that the only term containing $E_{-\alpha}$ is

$$\tilde{R}^{-1}a_0^{-1}(n_0^{-1}\partial_u n_0)a_0\tilde{R}. \tag{3.18}$$

Firstly, by analogy with the ansatz (2.14), we assume that

$$n_0^{-1}\partial_u n_0 = \sum_{\alpha\in\pi^+} \phi_\alpha E_{-\alpha}. \tag{3.19a}$$

that is, we impose the condition that, for $\alpha \notin \pi^+$,

$$\phi_\alpha = 0, \tag{3.20a}$$

and then (3.18) becomes

$$\begin{aligned}\tilde{R}^{-1}a_0^{-1}\sum_{\alpha\in\pi^+}\phi_\alpha E_{-\alpha}a_0\tilde{R} &= \tilde{R}^{-1}\sum_{\alpha\in\pi^+}\exp\Big(\sum_{\beta\in\pi^+}K_{\alpha\beta}\mu_\beta\Big)\phi_\alpha E_{-\alpha}\tilde{R}\\ &= \exp(\tilde{\psi}_\alpha H_\alpha)\sum_{\alpha\in\pi^+}\exp\Big(\sum_{\beta\in\pi^+}K_{\alpha\beta}\mu_\beta\Big)\phi_\alpha E_{-\alpha}\exp(-\tilde{\psi}_\alpha H_\alpha)\\ &= \sum_{\alpha\in\pi^+}\exp\Big(\sum_{\beta\in\pi^+}K_{\alpha\beta}(\mu_\beta-\tilde{\psi}_\beta)\Big)\phi_\alpha E_{-\alpha}.\end{aligned}$$

This last term should be compared with the corresponding term in $B_u = R^{-1}\partial_u R$; that is, from (2.28)

$$\sum_{\alpha\in\pi^+}\exp\Big(-\sum_{\beta\in\pi^+}K_{\alpha\beta}\tilde{\psi}_\beta\Big)E_{-\alpha}.$$

Hence

$$\phi_\alpha = \exp\Big(-\sum_{\beta\in\pi^+}K_{\alpha\beta}\mu_\beta\Big). \tag{3.21a}$$

Similarly, by considering $B_{\bar{u}}$ in the same way, we may show that

$$\tilde{\phi}_\alpha = \exp\Big(-\sum_{\beta\in\pi^+}K_{\alpha\beta}\tilde{\mu}_\beta\Big), \tag{3.21b}$$

where when α is a simple root $\tilde{\phi}_\alpha$ is defined by

$$\tilde{n}_0^{-1}\partial_{\bar{u}}\tilde{n}_0 = -\sum_{\alpha\in\pi^+}\tilde{\phi}_\alpha E_{+\alpha} \tag{3.19b}$$

and for $\alpha \notin \pi^+$

$$\tilde{\phi}_\alpha = 0. \tag{3.20b}$$

However, if we use the representations for n_0 and $\tilde{n}_0$ in (3.19), it is possible to show that, for $\alpha \in \pi^+$,

$$\phi_\alpha = \partial_u \nu_\alpha, \qquad \tilde{\phi}_\alpha = \partial_{\bar{u}}\tilde{\nu}_\alpha. \tag{3.22}$$

Hence (3.21) and (3.22) provide a relationship between the parameters ν_α and μ_α appearing on the right-hand side of (3.16) and, bearing this in mind, we shall return to the further simplification of that equation. By considering the explicit form of the matrices H_α for each algebra in appendix 1, we can reduce (3.16) to

$$\left.\begin{array}{ll}\exp(\psi_m+\tilde{\psi}_m)=\exp(\mu_m+\tilde{\mu}_m)\det(\tilde{n}_0^{-1}n_0)_m, & 1\leqslant m\leqslant n \text{ for } \mathfrak{a}_n, \mathfrak{c}_n,\\ & 1\leqslant m\leqslant n-2 \text{ for } \mathfrak{d}_n,\\ \exp(\psi_{m-1}+\tilde{\psi}_{m-1})=\exp(\mu_{m-1}+\tilde{\mu}_{m-1})\det(\tilde{n}_0^{-1}n_0)_m, & 2\leqslant m\leqslant n+1 \text{ for } \mathfrak{b}_n,\\ \exp(\psi_{n-1}+\tilde{\psi}_{n-1})=\exp(\mu_{n-1}+\tilde{\mu}_{n-1})[\det(\tilde{n}_0^{-1}n_0)_{n-1}/\det(\tilde{n}_0^{-1}n_0)_n]^{1/2}, & \\ & \text{for } \mathfrak{d}_n\\ \exp(\psi_n+\tilde{\psi}_n)=\exp(\mu_n+\tilde{\mu}_n)[\det(\tilde{n}_0^{-1}n_0)_{n-1}\det(\tilde{n}_0^{-1}n_0)_n]^{1/2}, & \end{array}\right\} \quad (3.23)$$

where the abbreviation $\det(\tilde{n}_0^{-1}n_0)_m$ refers to the determinant of the submatrix $[(\tilde{n}_0^{-1}n_0)_{kl}]$ with k and l in the appropriate range.

The parameters in the exponent on the left-hand side of the equations (3.23) are just those which satisfy the Toda lattice equations (2.26). In fact, by taking logarithms of each side of (3.23) and then substituting for $\tilde{n}_0$ and n_0 the representations (3.12) subjected to the conditions (3.19)–(3.22), we obtain a parametrisation for the solutions of the Toda lattice equation. These comments shall be exemplified below by referring to some specific groups.

In all the examples below, we shall use the representations for $E_{-\alpha}$ and $E_{+\alpha}$ given in appendix 1. For those involving an SU($n+1$) group the notation will be that of appendix 2.

(i) SU(2). There is only one root and so, with

$$n_0=\exp(\nu_{21}e_{21}), \qquad \tilde{n}_0=\exp(-\tilde{\nu}_{12}e_{12}),$$

$(\tilde{n}_0^{-1}n_0)$ is simply given by

$$\begin{pmatrix}1+\tilde{\nu}_{12}\nu_{21} & \tilde{\nu}_{12}\\ \nu_{21} & 1\end{pmatrix}.$$

Also

$$\phi=\partial_u\nu_{21}=\mathrm{e}^{-2\mu}, \qquad \tilde{\phi}=\partial_{\bar{u}}\tilde{\nu}_{12}=\mathrm{e}^{-2\tilde{\mu}},$$

so (3.23) gives

$$\mathrm{e}^{(\psi+\tilde{\psi})}=[(\partial_u\nu_{21})(\partial_{\bar{u}}\tilde{\nu}_{12})]^{-1/2}(1+\tilde{\nu}_{12}\nu_{21}). \quad (3.24)$$

If we use the variable

$$\rho=-(\psi+\tilde{\psi}),$$

then (3.24) is

$$\rho=-\ln(1+\tilde{\nu}_{12}\nu_{21})+\tfrac{1}{2}\ln(\partial_u\nu_{21}\,\partial_{\bar{u}}\tilde{\nu}_{12}),$$

reproducing Witten's SU(2) solution (Witten 1977).

(ii) SU(3).

$$n_0=\exp(\nu_{ij}e_{ij})=1+\nu_{ij}e_{ij}+\tfrac{1}{2}\delta_{mi}e_{kj}\nu_{ij}\nu_{km}$$

and so

$$\phi_1=\partial_u\nu_{21}=\exp(-2\mu_1+\mu_2), \qquad \phi_2=\partial_u\nu_{32}=\exp(\mu_1-2\mu_2).$$

There is also the additional constraint

$$0=\phi_3=\partial_u\nu_{31}+\tfrac{1}{2}(\partial_u\nu_{21})\nu_{32}-\tfrac{1}{2}\nu_{21}(\partial_u\nu_{32}). \quad (3.25a)$$

Similarly,

$$\tilde{\phi}_1 = \partial_{\bar{u}}\tilde{\nu}_{12} = \exp(-2\tilde{\mu}_1 + \tilde{\mu}_2), \qquad \tilde{\phi}_2 = \partial_{\bar{u}}\tilde{\nu}_{23} = \exp(\tilde{\mu}_1 - 2\tilde{\mu}_2),$$
$$0 = \tilde{\phi}_3 = \partial_{\bar{u}}\tilde{\nu}_{13} - \tfrac{1}{2}(\partial_{\bar{u}}\tilde{\nu}_{12})\tilde{\nu}_{23} + \tfrac{1}{2}\tilde{\nu}_{12}(\partial_{\bar{u}}\tilde{\nu}_{23}). \tag{3.25b}$$

If we denote $\tilde{n}_0^{-1}n_0$ by $\mathcal{N}$ then

$$\mathcal{N}_{11} = 1 + \tilde{\nu}_{12}\nu_{21} + (\tilde{\nu}_{13} + \tfrac{1}{2}\tilde{\nu}_{12}\tilde{\nu}_{23})(\nu_{31} + \tfrac{1}{2}\nu_{32}\nu_{21}),$$
$$\mathcal{N}_{22} = 1 + \tilde{\nu}_{23}\nu_{32},$$
$$\mathcal{N}_{12} = \tilde{\nu}_{12} + \nu_{32}(\tilde{\nu}_{13} + \tfrac{1}{2}\tilde{\nu}_{12}\tilde{\nu}_{23}),$$
$$\mathcal{N}_{21} = \nu_{21} + \tilde{\nu}_{23}(\nu_{31} + \tfrac{1}{2}\nu_{32}\nu_{21}).$$

So using (3.23)

$$\exp(\psi_1 + \tilde{\psi}_1) = [(\partial_u\nu_{21})(\partial_{\bar{u}}\tilde{\nu}_{12})]^{-2/3}[(\partial_u\nu_{32})(\partial_{\bar{u}}\tilde{\nu}_{23})]^{-1/3}$$
$$\times[1 + \tilde{\nu}_{12}\nu_{21} + (\tilde{\nu}_{13} + \tfrac{1}{2}\tilde{\nu}_{12}\tilde{\nu}_{23})(\nu_{31} + \tfrac{1}{2}\nu_{32}\nu_{21})], \tag{3.26a}$$

$$\exp(\psi_2 + \tilde{\psi}_2) = [(\partial_u\nu_{21})(\partial_{\bar{u}}\tilde{\nu}_{12})]^{-1/3}[(\partial_u\nu_{32})(\partial_{\bar{u}}\tilde{\nu}_{23})]^{-2/3}$$
$$\times[1 + \tilde{\nu}_{23}\nu_{32} + (\tilde{\nu}_{13} - \tfrac{1}{2}\tilde{\nu}_{12}\tilde{\nu}_{23})(\nu_{31} - \tfrac{1}{2}\nu_{32}\nu_{21})]. \tag{3.26b}$$

(iii) Sp(2). The first example with R and $\tilde{R}$ non-triangular.
From appendix 1,

$$E_{-1} = e_{21} - e_{34}, \qquad E_{-2} = e_{42}, \qquad E_{-3} = e_{31}, \qquad E_{-4} = e_{32} + e_{41}.$$

Again denoting

$$\tilde{n}_0^{-1}n_0 = \exp(\tilde{\nu}_\alpha E_\alpha)\exp(\nu_\alpha E_{-\alpha})$$

by $\mathcal{N}$, we have

$$\mathcal{N}_{11} = 1 + \nu_1\tilde{\nu}_1 + V_3\tilde{V}_3 + V_4^+\tilde{V}_4^+, \qquad \mathcal{N}_{12} = \tilde{\nu}_1 + \tilde{V}_3 V_4^- + \nu_2\tilde{V}_4^+,$$
$$\mathcal{N}_{22} = 1 + V_4^-\tilde{V}_4^- + \nu_2\tilde{\nu}_2, \qquad \mathcal{N}_{21} = \nu_1 + V_3\tilde{V}_4^- + \tilde{\nu}_2 V_4^+;$$

the other elements of the matrix need not be specified and we have used the abbreviations

$$V_4^\pm = \nu_4 \pm \tfrac{1}{2}\nu_1\nu_2, \qquad \tilde{V}_4^\pm = \tilde{\nu}_4 \pm \tfrac{1}{2}\tilde{\nu}_1\tilde{\nu}_2,$$
$$V_3 = \nu_3 - \tfrac{1}{6}\nu_1^2\nu_2, \qquad \tilde{V}_3 = \tilde{\nu}_3 - \tfrac{1}{6}\tilde{\nu}_1^2\tilde{\nu}_2.$$

Also the parameters are related by the equations

$$\phi_1 = \partial_u\nu_1 = \exp(-2\mu_1 + \mu_2), \qquad \phi_2 = \partial_u\nu_2 = \exp(-2\mu_2 + 2\mu_1), \tag{3.27}$$

together with similar equations for $\tilde{\phi}_1$ and $\tilde{\phi}_2$. The constraints $\phi_3 = \phi_4 = 0$, $\tilde{\phi}_3 = \tilde{\phi}_4 = 0$ reduce to respectively

$$\partial_u V_3 - V_4^-(\partial_u\nu_1) = 0, \qquad \partial_u V_4^+ - \nu_2\partial_u\nu_1 = 0, \tag{3.28a}$$
$$\partial_u\tilde{V}_3 - \tilde{V}_4^-(\partial_u\tilde{\nu}_1) = 0, \qquad \partial_u\tilde{V}_4^+ - \tilde{\nu}_2\partial_u\tilde{\nu}_1 = 0. \tag{3.28b}$$

By straightforward substitution of $\det\mathcal{N}$ and the relations (3.27) into (3.23), the solutions are given by

$$\exp(\psi_1 + \tilde{\psi}_1) = [(\partial_u\nu_1)(\partial_{\bar{u}}\tilde{\nu}_1)]^{-1}[(\partial_u\nu_2)(\partial_{\bar{u}}\tilde{\nu}_2)]^{-1/2}[1 + \nu_1\tilde{\nu}_1 + V_3\tilde{V}_3 + V_4^+\tilde{V}_4^+], \tag{3.29a}$$

$$\exp(\psi_2+\tilde{\psi}_2)=[(\partial_u\nu_1)(\partial_{\bar{u}}\tilde{\nu}_1)]^{-1}[(\partial_u\nu_2)(\partial_{\bar{u}}\tilde{\nu}_2)]^{-1}$$
$$\times[(1+\nu_1\tilde{\nu}_1+V_3\tilde{V}_3+V_4^+\tilde{V}_4^+)(1+\nu_2\tilde{\nu}_2+V_4^-\tilde{V}_4^-)$$
$$-(\nu_1+\tilde{\nu}_2V_4^++V_3\tilde{V}_4^-)(\tilde{\nu}_1+\nu_2\tilde{V}_4^++\tilde{V}_3V_4^-)]. \tag{3.29b}$$

Of course, the solutions are more accurately expressed by introducing the explicit forms for V_3, $\tilde{V}_3$, $V_4^{\pm}$ and $\tilde{V}_4^{\pm}$. However, the form (3.29) is often more convenient to manipulate; for example, to prove that (3.29) really do satisfy the Toda lattice equation. This is also where the constraints (3.28) come into play. To demonstrate this, we calculate $\partial_u\partial_{\bar{u}}(\psi_1+\tilde{\psi}_1)$ by taking logarithms of (3.29a) and then differentiating:

$$\partial_u\partial_{\bar{u}}(\psi_1+\tilde{\psi}_1)=\Psi(1+\nu_1\tilde{\nu}_1+V_3\tilde{V}_3+V_4^+\tilde{V}_4^+)^{-2}$$
$$=\Psi[(\partial_u\nu_1)(\partial_{\bar{u}}\tilde{\nu}_1)]^{-2}[(\partial_u\nu_2)(\partial_{\bar{u}}\tilde{\nu}_2)]^{-1}\exp[-2(\psi_1+\tilde{\psi}_1)]. \tag{3.30}$$

Ψ is used to denote

$$(1+\nu_1\tilde{\nu}_1+V_3\tilde{V}_3+V_4^+\tilde{V}_4^+)(\partial_u\nu_1\partial_{\bar{u}}\tilde{\nu}_1+\partial_uV_3\partial_{\bar{u}}\tilde{V}_3+\partial_uV_4^+\partial_{\bar{u}}\tilde{V}_4^+)$$
$$-(\nu_1\partial_{\bar{u}}\tilde{\nu}_1+V_3\partial_{\bar{u}}\tilde{V}_3+V_4^+\partial_{\bar{u}}\tilde{V}_4^+)(\tilde{\nu}_1\partial_u\nu_1+\tilde{V}_3\partial_uV_3+\tilde{V}_4^+\partial_uV_4^+)$$
$$=(\partial_u\nu_1)(\partial_{\bar{u}}\tilde{\nu}_1)[(1+\nu_1\tilde{\nu}_1+V_3\tilde{V}_3+V_4^+\tilde{V}_4^+)(1+V_4^-\tilde{V}_4^-+\nu_2\tilde{\nu}_2)$$
$$-(\nu_1+\tilde{V}_4^-V_3+V_4^+\tilde{\nu}_2)(\tilde{\nu}_1+\tilde{V}_3V_4^-+\tilde{V}_4^+\nu_2)]$$
$$\equiv[(\partial_u\nu_1)(\partial_{\bar{u}}\tilde{\nu}_1)]^2[(\partial_u\nu_2)(\partial_{\bar{u}}\tilde{\nu}_2)]\exp(\psi_2+\tilde{\psi}_2),$$

using (3.28) and (3.29b). So (3.30) becomes

$$\partial_{\bar{u}}\partial_u(\psi_1+\tilde{\psi}_1)=\exp[(\psi_2+\tilde{\psi}_2)-2(\psi_1+\tilde{\psi}_1)]$$

and comparison of this with (2.26) shows that $\psi_1+\tilde{\psi}_1$ given by (3.29a) indeed satisfies the Toda lattice equation.

(iv) SU(4). We shall briefly consider the main features of the solutions. It is illustrative of the general SU($n+1$) solution, since unlike SU(2) and SU(3), but similar to Sp(2), the original form of the constraints (3.20) is rearranged. In this way the derivatives of coefficients of the $E_{-\alpha}$ in n_0 corresponding to non-simple roots are expressed as the derivatives of coefficients of those for simple roots. Here, if

$$n_0=\exp(\nu_{ij}e_{ij}),$$

then the form (3.20a) of the constraints are

$$\partial_uV_{31}-\nu_{32}(\partial_u\nu_{21})=0,\qquad \partial_uV_{42}-\nu_{43}(\partial_u\nu_{32})=0,$$
$$\partial_uV_{41}-\nu_{43}(\partial_uV_{31})-V_{42}\partial_u\nu_{21}=0,$$

where

$$V_{31}=\nu_{31}+\tfrac{1}{2}\nu_{32}\nu_{21},\qquad V_{42}=\nu_{42}+\tfrac{1}{2}\nu_{43}\nu_{32},$$
$$V_{41}=\nu_{41}+\tfrac{1}{2}(\nu_{42}\nu_{21}+\nu_{43}\nu_{31}).$$

The first two constraint equations have a convenient form and, using the first, the third can be reduced to

$$\partial_uV_{41}-V_{42}\partial_u\nu_{21}=0,$$

as required. The value of this rearrangement was shown for Sp(2) since it provides for easy manipulation of Ψ in (3.30) to give a common factor $(\partial_u \nu_1)(\partial_{\bar{u}} \tilde{\nu}_1)$. There are similar conditions for the tilde parameters.

The general parametrisation for the SU($n+1$) solution is given in appendix 2, so we shall not consider further the SU(4) case.

(v) SO(5) and SO(6). Since the algebras $\mathfrak{b}_2$ and $\mathfrak{c}_2$ are isomorphic, the solutions of the Toda lattice equations governed by SO(5) are the same as those for Sp(2). The Cartan matrix for SO(5) is the transpose of that for Sp(2), so if we exchange the parameters ψ_1 and ψ_2 in (3.29) we have solutions (3.23) for SO(5). Also, $\mathfrak{a}_3$ and $\mathfrak{d}_3$ are isomorphic and similar remarks hold for their Cartan matrices. So the solutions for SO(6) are given by those for SU(4) with ψ_3 unchanged and ψ_1 and ψ_2 interchanged.

4. Discussion and concluding remarks

In §§ 2 and 3 we have applied the Iwasawa decomposition to the complexification of some semi-simple Lie group G. By this method we have been able not only to derive the Toda lattice equations governed by G, but also to parametrise the solutions to the equations.

The derivation of the equations is based on a generalisation of Yang's R-gauge method. However, unlike Yang, we have not imposed any relation between the parameters of R and $\tilde{R}$. This is possible since we are always able to solve the subsidiary equations for z_α and $\tilde{z}_\alpha$ in terms of ψ_α and $\tilde{\psi}_\alpha$, and the latter pair of parameters always appear in the combination

$$\psi_\alpha + \tilde{\psi}_\alpha = -(K^{-1})_{\alpha\beta}\rho_\beta.$$

However, if we wish to take a more physical viewpoint, then we should be able to recover the original gauge theory when we transform back to the real coordinates, s and t. In this case we must demand that B_u and $B_{\bar{u}}$ have values in the Lie algebra of G and not G_c. Consequently, we have to impose some condition on B_u and $B_{\bar{u}}$ and hence impose some relationship between R and $\tilde{R}$. The condition is constructed by invoking the differences between the Lie algebras of G and G_c. We show that a sufficient but not necessary condition is given by

$$\tilde{R} = (R^\dagger)^{-1}, \tag{4.1}$$

not only for G = SU($n+1$) (Yates 1978), but also when G is any classical group.

The complex extensions G_c must first be divided into two categories to state explicitly the differences between G and G_c. Category (i) contains SL($n+1$, $\mathbb{C}$) and Sp(n, $\mathbb{C}$), whilst (ii) is comprised of SO($2n+1$, $\mathbb{C}$), SO($2n$, $\mathbb{C}$). For those in (i) the elements of G are unitary and in (ii) the elements of G are real. These properties are not exhibited by the corresponding G_c and so the respective conditions on the potentials are

$$\text{(i)}\quad B_u^\dagger = -B_{\bar{u}}, \tag{4.2a}$$

$$\text{(ii)}\quad B_u^* = B_{\bar{u}}. \tag{4.2b}$$

These conditions are satisfied if (4.1) holds for elements R, $\tilde{R} \in G_c/G$ in both categories. Since

$$B_{\bar{u}} = R^\dagger(\partial_{\bar{u}}(R^\dagger)^{-1}) = -[R^{-1}\partial_u R]^\dagger = -B_u^\dagger$$

(4.2a) is satisfied. Also for category (ii) the elements of G_c and G are orthogonal, so

$$(R^{\dagger})^{-1} = (R^{\mathrm{T}})^{\dagger} = R^*$$

and (4.1) becomes

$$\tilde{R} = R^*,$$

which ensures that (4.2b) is satisfied.

Since the potentials are left-invariant the condition (4.1) is only sufficient for (4.2) to be satisfied. We may left translate R and $\tilde{R}$, so that the transformed elements are no longer related by (4.1), but (4.2) still remains true.

Using the representations (2.6) and (2.9), the condition (4.1) reduces to a relationship between the parameters of R and $\tilde{R}$, namely

$$\tilde{z}_\alpha = \bar{z}_\alpha, \qquad \tilde{\psi}_\alpha = \psi_\alpha. \tag{4.3}$$

A second notable feature of the method here is the introduction of the factors $r(u)$ and $\tilde{r}(\bar{u})$ to obtain explicit expressions for the solutions. This feature is not paralleled by the one-dimensional model, obviously since the functions are only dependent on the one coordinate, t. However, the algebraic properties of the one-dimensional model of Olshanetsky and Perelomov (1979) do have some similarity to the two-dimensional case. If we wish to compare our method with that of Olshanetsky and Perelomov (hereafter referred to as OP), we must relate r and $\tilde{r}$ to some factor in the one-dimensional model.

To achieve this, we first reduce u and $\bar{u}$ to t, so that (1.18) becomes

$$\dot{A} - \dot{B} = [A, B] \tag{4.4}$$

with $A = R^{-1}\dot{R}$ and $B = \tilde{R}^{-1}\dot{\tilde{R}}$, the dot referring to differentiation with respect to t. If we introduce

$$L = B - A, \tag{4.5}$$

then (4.4) has the form of a Lax pair

$$\dot{L} = [L, A]. \tag{4.6}$$

Drawing on the theory of the Lax pair, we can say that (4.6) is equivalent to

$$L(t) = R^{-1}L(0)R, \qquad A = R^{-1}\dot{R}. \tag{4.7}$$

By rearranging (4.7) and then using (4.5), we obtain

$$L(0) = RL(t)R^{-1} = R\tilde{R}^{-1}\dot{\tilde{R}}R^{-1} - \dot{R}R^{-1} = -\partial_t(R\tilde{R}^{-1})\,(R\tilde{R}^{-1})^{-1},$$

which becomes

$$L(0) = -\dot{x}x^{-1}, \tag{4.8}$$

when we define

$$x = R\tilde{R}^{-1}. \tag{4.9}$$

Substitution of the decompositions (2.11) for R and $\tilde{R}$ in (4.9) gives

$$x = na\tilde{a}^{-1}\tilde{n}^{-1} \equiv nh\tilde{n}^{-1}, \tag{4.10}$$

where $h = a\tilde{a}^{-1}$ is a diagonal matrix. (4.10) exactly reproduces the results in OP. x is described by the initial data since it follows from (4.8) that

$$x(t) = x(0)\exp[-L(0)t]. \tag{4.11}$$

From (4.9) and (3.13) we observe that $x(t)$ corresponds to $\tilde{r}_0^{-1}r_0$ and also in (4.10) h contains the Toda lattice variables. So the matrix manipulations on $x(t)$ used in OP to solve the equations are equivalent to those we apply to (3.13), also to solve the equations.

There may be a fundamental reason mathematically for splitting up x into the product of functions (4.9). Such an idea deserves more thought, particularly with reference to the theory of the symmetric space G_c/G to which R and $\tilde{R}$ belong.

In § 2 the Toda lattice equations are derived from the condition that the gauge field strength $F_{u\bar{u}}$ should be sourceless, and in § 3 this condition was shown to imply that the curvature two-form (3.2) should vanish. Now (3.2) may be considered as the integrability condition for the set of linear equations

$$\mathrm{d}\theta = \theta\omega, \tag{4.12}$$

where θ is a row of 0-forms. The set of equations (4.12) should be investigated for our model, since it is anticipated that they might provide some insight into the Bäcklund transformations of the Toda lattice equations (Farwell and Minami 1982).

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Appendix 1. A matrix representation for the Chevalley basis

We shall use e_{ij}, $1 \leq i, j \leq n$, to denote the $n \times n$ matrix in which the only non-zero entry is 1, where the ith row and jth column intersect (Humphreys 1972). From this definition it follows that

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \tag{A1.1}$$

and so

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}. \tag{A1.2}$$

In each case below the suffix i on the basis elements H_i of $\mathfrak{h}$ refers to some specific ordering of the simple roots comprising π^+. By E_{+i} and E_{-i}, we mean the corresponding elements of the bases of $\mathfrak{g}_{+\alpha}$ and $\mathfrak{g}_{-\alpha}$ respectively; that is, those which satisfy the commutation relations (2.2c), namely

$$[E_{+i}, E_{-i}] = H_i.$$

The required E_{+i}, E_{-i} may be selected from the sets $\{E_{+\alpha}\}$, $\{E_{-\alpha}\}$ by inspection of the commutation relations derived using (A1.2).

Note that in the following representations for each class of Lie algebra, the matrix E_{+i} is taken to be the transpose of E_{-i}. Such a specification conforms with the commutation relations (2.2).

(i) $\mathfrak{a}_n$—$(n+1)\times(n+1)$ matrix representation

Basis for $\mathfrak{h}$	$H_i = e_{ii} - e_{i+1i+1}$	$(1 \le i \le n)$
$\mathfrak{g}_{-\alpha,\alpha\in\Delta^+}$	e_{ij}	$(1 \le j < i \le n+1)$
$\mathfrak{g}_{+\alpha,\alpha\in\Delta^+}$	$e_{ji} = e_{ij}^{\mathrm{T}}$	$(1 \le j < i \le n+1)$.

(ii) $\mathfrak{b}_n$—$(2n+1)\times(2n+1)$ matrix representation

Basis for $\mathfrak{h}$	$H_{i-1} = e_{ii} - e_{i+1i+1} - e_{n+in+i} + e_{n+i+1n+i+1}$	$(2 \le i \le n)$
	$H_n = e_{n+1n+1} - e_{2n+12n+1}$	
$\mathfrak{g}_{-\alpha,\alpha\in\Delta^+}$	$e_{i+1j+1} - e_{j+n+1i+n+1}$	$(1 \le j < i \le n)$
	$e_{i+1+n1} - e_{1i+1}$	$(1 \le i \le n)$
	$e_{n+i+1j+1} - e_{n+j+1i+1}$	$(1 \le j < i \le n)$
$\mathfrak{g}_{+\alpha,\alpha\in\Delta^+}$	$e_{j+1i+1} - e_{i+n+1j+n+1}$	$(1 \le j < i \le n)$
	$e_{1i+1+n} - e_{i+11}$	$(1 \le i \le n)$.
	$e_{j+1n+i+1} - e_{i+1n+j+1}$	$(1 \le j < i \le n)$.

(iii) $\mathfrak{c}_n$—$2n\times 2n$ matrix representation

Basis for $\mathfrak{h}$	$H_i = e_{ii} - e_{i+1i+1} - e_{n+in+i} + e_{n+i+1n+i+1}$	$(1 \le i \le n-1)$
	$H_n = e_{nn} - e_{2n2n}$	
$\mathfrak{g}_{-\alpha,\alpha\in\Delta^+}$	$e_{ij} - e_{j+ni+n}$	$(1 \le j < i \le n)$
	e_{n+ii}	$(1 \le i \le n)$
	$e_{n+ji} + e_{n+ij}$	$(1 \le j < i \le n)$
$\mathfrak{g}_{+\alpha,\alpha\in\Delta^+}$	$e_{ji} - e_{n+ij+n}$	$(1 \le j < i \le n)$
	e_{in+i}	$(1 \le i \le n)$
	$e_{in+j} + e_{jn+i}$	$(1 \le j < i \le n)$.

(iv) $\mathfrak{d}_n$—$2n\times 2n$ matrix representation

Basis for $\mathfrak{h}$	$H_i = e_{ii} - e_{i+1i+1} - e_{n+in+i} + e_{n+i+1n+i+1}$	$(1 \le i \le n-1)$
	$H_n = e_{n-1n-1} + e_{nn} - e_{2n-12n-1} - e_{2n2n}$	
$\mathfrak{g}_{-\alpha,\alpha\in\Delta^+}$	$e_{ij} - e_{j+ni+n}$	$(1 \le j < i \le n)$
	$e_{n+ij} - e_{n+ji}$	$(1 \le j < i \le n)$
$\mathfrak{g}_{+\alpha,\alpha\in\Delta^+}$	$e_{ji} - e_{i+nj+n}$	$(1 \le j < i \le n)$
	$e_{jn+i} - e_{n+ji}$	$(1 \le j < i \le n)$.

It is important to note various properties of the bases for each class. In all cases, the H_i are traceless diagonal matrices. Furthermore, for $\mathfrak{a}_n$ the set $\{E_{-\alpha}\}$ is composed of lower triangular matrices with zeros on the diagonal, and so the transposed set $\{E_{+\alpha}\}$ consists of upper triangular matrices with no diagonal entries. Although the sets $\{E_{-\alpha}\}$

are not triangular for the other three classes, it is possible to select submatrices which do have this property. Let $[(E_{-\alpha})_{lm}]$ denote a submatrix of the basis element $E_{-\alpha}$ where the range of l and m is specified. Then

(a) for $\mathfrak{b}_n$: the set $\{[(E_{-\alpha})_{lm}]_{2\leq l,m\leq n+1}\}$ are lower triangular with zero diagonal entries and the set $\{[(E_{-\alpha})_{lm}]_{2\leq l\leq n+1, m=1 \text{ or } n+2\leq m\leq 2n+1}\}$ have all zero entries;

(b) for $\mathfrak{c}_n$ and $\mathfrak{d}_n$: the set $\{[(E_{-\alpha})_{lm}]_{1\leq l,m\leq n}\}$ are lower triangular with zero diagonal entries and the set $\{[(E_{-\alpha})_{lm}]_{1\leq l\leq n, n+1\leq m\leq 2n}\}$ have all zero entries.

Corresponding results concerning upper diagonal and zero submatrices of $E_{+\alpha}$ may be deduced using the operation of transposition.

Appendix 2. Parametrisation of the SU($n+1$) solution

In § 3 we show that the solution of the Toda lattice equation governed by SU($n+1$) is given by

$$\psi_m + \tilde{\psi}_m = \mu_m + \tilde{\mu}_m + \ln \det(\tilde{n}_0^{-1} n_0)_m, \qquad 1 \leq m \leq n. \tag{A2.1}$$

In general, we use the representations (3.12) for n_0 and $\tilde{n}_0$. However, since in appendix 1, for $\mathfrak{a}_n$ each $E_{-\alpha}$ has the simple form e_{ij}, $1\leq j<i\leq n+1$, we adopt the double index notation for ν and $\tilde{\nu}$ and write

$$\begin{aligned} n_0 &= \exp\Big(\sum_{i>j} \nu_{ij} e_{ij}\Big) \\ &= 1 + \sum_{1\leq j<i\leq n+1} \nu_{ij} e_{ij} + \sum_{1\leq j<k<i\leq n+1} \nu_{ik}\nu_{kj} e_{ij}, \end{aligned} \tag{A2.2}$$

$$\begin{aligned} \tilde{n}_0^{-1} &= \exp\Big(\sum_{i>j} \tilde{\nu}_{ji} e_{ji}\Big) \\ &= 1 + \sum_{1\leq j<i\leq n+1} \tilde{\nu}_{ji} e_{ji} + \sum_{1\leq j<k<i\leq n+1} \tilde{\nu}_{jk}\tilde{\nu}_{ki} e_{ji}. \end{aligned} \tag{A2.3}$$

We use the abbreviations

$$V_{ij}^{\pm} = \nu_{ij} \pm \sum_{k=j+1}^{i-1} \nu_{ik}\nu_{kj}, \qquad \tilde{V}_{ji}^{\pm} = \tilde{\nu}_{ji} \pm \sum_{k=j+1}^{i-1} \tilde{\nu}_{jk}\tilde{\nu}_{ki}. \tag{A2.4}$$

The range of summation in the second terms ensures that corresponding to the simple roots, $i=j+1$,

$$V_{ij}^{\pm} \equiv \nu_{ij}, \qquad \tilde{V}_{ji}^{\pm} \equiv \tilde{\nu}_{ji}.$$

Then (A2.2), (A2.3) become

$$n_0 = 1 + \sum_{1\leq j<i\leq n+1} V_{ij}^{+} e_{ij}, \qquad \tilde{n}_0^{-1} = 1 + \sum_{1\leq j<i\leq n+1} \tilde{V}_{ji}^{+} e_{ji}. \tag{A2.5}$$

Conditions on V_{ij} and $\tilde{V}_{ji}$ arise as a consequence of the constraints (3.20):

$$\partial_u V_{ij}^{+} - V_{ik}^{-}(\partial_u V_{kj}^{+}) = 0 \qquad \text{for } i>j+1, \tag{A2.6}$$

$$\partial_{\bar{u}} \tilde{V}_{ji}^{-} + \tilde{V}_{jk}^{+}(\partial_{\bar{u}} \tilde{V}_{ki}^{-}) = 0 \qquad \text{for } i>j+1. \tag{A2.7}$$

Now, from (A2.5)

$$\tilde{n}_0^{-1}n_0 = 1 + \sum_{1\le i<k\le n+1} \tilde{V}^+_{ik}V^+_{ki}e_{ii} + \sum_{1\le j<i\le n+1}\left(\tilde{V}^+_{ji} + \sum_{k=j+1}^{i-1}\tilde{V}^+_{jk}V^+_{ki}\right)e_{ji}$$
$$+ \sum_{1\le j<i\le n+1}\left(V^+_{ij} + \sum_{k=j+1}^{i-1}V^+_{ik}\tilde{V}^+_{kj}\right)e_{ij}$$

and so the submatrix

$$((\tilde{n}_0^{-1}n_0)_{pq})_{1\le p,q\le m<n} = I_m + \sum_{i=1}^{m}\sum_{k=i+1}^{n+1}\tilde{V}^+_{ik}V^+_{ki}e_{ii}$$
$$+ \sum_{1\le j<i\le m}\left(\tilde{V}^+_{ji} + \sum_{k=j+1}^{i-1}\tilde{V}^+_{jk}V^+_{ki}\right)e_{ji} + \sum_{1\le j<i\le m}\left(V^+_{ij} + \sum_{k=j+1}^{i-1}V^+_{ik}\tilde{V}^+_{kj}\right)e_{ij}. \tag{A2.8}$$

The relations (3.21) and (3.22) become for each $j = 1, 2, \ldots, n$

$$(\partial_u V_{j+1j})(\partial_{\bar{u}}\tilde{V}_{jj+1}) = \exp\left(-\sum_{m=1}^{n}K_{jm}(\mu_m + \tilde{\mu}_m)\right), \tag{A2.9}$$

where (K_{jm}): $j, m = 1, 2, \ldots, n$ is the Cartan matrix for SU$(n+1)$.

If we define the variable

$$\rho_j = -K_{jm}(\psi_m + \tilde{\psi}_m), \tag{A2.10}$$

then the solution (A2.1) becomes for $j = 1, 2, \ldots, n$,

$$\rho_j = \ln[(\partial_u \nu_{j+1j})(\partial_{\bar{u}}\tilde{\nu}_{j+1j})] + \ln\left(\frac{\det(\tilde{n}_0^{-1}n_0)_{j+1}\det(\tilde{n}_0^{-1}n_0)_{j-1}}{[\det(\tilde{n}_0^{-1}n_0)_j]^2}\right) \tag{A2.11}$$

with $(\tilde{n}_0^{-1}n_0)_m$ given by (A2.8), with the proviso that the parameters are subjected to (A2.6) and (A2.7).

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